J. DIFFERENTIAL GEOMETRY
17 (1982) 523–529

CURVATURE OF AN ∞-DIMENSIONAL MANIFOLD RELATED TO HILL'S EQUATION

H. P. McKEAN

1. Introduction

Let C_+^∞ be the space of positive infinitely differentiable functions e_0 of period 1 with $\int_0^1 e_0^2 = 1$, and let M be the class of real infinitely differentiable functions q of period 1 such that the corresponding Hill's operator $Q = -D^2 + q$ has ground state $\lambda_0 = 0$, where D signifies differentiation with regard to $0 \leqslant x < 1$. The map $C_+^\infty \to M$ defined by $e_0''/e_0 = q$ is $1:1$ and onto, the ground state of Q being necessarily simple; in particular, M comes in one simply-connected piece. The purpose of this note is to study the curvature of M considered as immersed in the space C_1^∞ of all real infinitely differentiable functions of period 1; evidently, it is a surface of codimension 1 defined by the single relation $\lambda_0 = 0$, and since the gradient of the latter is $\nabla\lambda_0 = e_0^2 \neq 0$, M sits smoothly in C_1^∞.

The curvatures of 2-dimensional slices of M are found to be positive, the principal curvatures being proportional to the reciprocals of the excited periodic eigenvalues $0 < \bar{\lambda}_j$ $(j = 1, 2, 3, \cdots)$ of the so-called *allied operator* $\bar{Q}$. The latter is the Hill's operator with ground state proportional to $e_0^{3/2}$ relative to the scale $d\bar{x} = (\int_0^1 e_0)^{-1} e_0\, dx$. The *maximal curvature* of a 2-dimensional slice is

$$m = 4\left(\int_0^1 e_0\right)^4 \left(\int_0^1 e_0^4\right)^{-1} \times (\bar{\lambda}_1\bar{\lambda}_2)^{-1},$$

while the *total curvature* is

$$k = 4\left(\int_0^1 e_0\right)^4 \left(\int_0^1 e_0^4\right)^{-1} \times \sum_{1 \leqslant i < j} (\bar{\lambda}_i\bar{\lambda}_j)^{-1}.$$

Received June 23, 1981. The support of the National Science Foundation under Grant NSF-MCS-76-0739 is gratefully acknowledged. Reproduction in whole or in part is permitted for any purpose of the U.S. Govt.

The latter may be expressed directly in terms of the ground state e_0:

$$k = 4\left(\int_0^1 e_0^{-2}\right)^2\left(\int_0^1 e_0^4\right)^{-2}\int\int\int e_0^4(x_1)e_0^4(x_2)e_0^4(x_3)$$
$$\times\int_{x_1^*}^{x_2^*} e_0^{-2}\int_{x_2^*}^{x_3^*} e_0^{-2}\int_{x_3^*}^{x_1^*} e_0^{-2}\, d^3x,$$

in which x_1^*, x_2^*, x_3^* are the points x_1, x_2, x_3 arranged in their natural order around the circle. For example, at the place $q = 0$, $m = \frac{1}{4}\pi^{-4}$ and $k = 1/90$. The quantities m and k may be as large or as small as one pleases; for e_0 approximating $x^{-1/4}$ $(0 \leqslant x < 1)$, k is small, while for e_0^2 approximating a saw-tooth function of period $1/3$, m is large: in the first case, the potential approximates $(5/20)x^{-2}$, while in the second it has 6 poles of alternating signature.

A manifold M of different character is obtained by fixing the first excited eigenvalue of Q at $\lambda_1 = 0$, say. M comprises the functions q of class C_1^∞ expressible as e_1''/e_1, the function e_1 having just 2 simple roots per period. This is a more complicated manifold exhibiting some negative curvature; in fact, the second fundamental form has just one negative eigenvalue. The computations are similar and readily extended to the higher eigenvalues λ_2, λ_3, *etc.*

2. The second fundamental form

Let $e_n (n \geqslant 0)$ be the full set of periodic eigenfunctions of Q corresponding to the eigenvalues $\lambda_0 < \lambda_1 \leqslant \lambda_2 < \lambda_3 \leqslant \lambda_4 < etc.$ The unit normal to M at q is $n = (\int_0^1 e_0^4)^{-1/2}e_0^2$, and with the aid of the inverse operator

$$Q^{-1}: f \to \sum_{n=1}^{\infty} \lambda_n^{-1} e_n(f, e_n) = \int_0^1 Q_{xy}^{-1} f(y)\, dy,$$

mapping the annihilator of e_0 into itself, it is a simple matter to compute

$$\frac{\partial e_0(x)}{\partial q(y)} = -Q_{xy}^{-1}e_0(y),$$
$$\frac{\partial n(x)}{\partial q(y)} = -2\left(\int_0^1 e_0^4\right)^{-1/2} e_0(x)Q_{xy}^{-1}e_0(y) + 2\left(\int e_0^4\right)^{-3/2} e_0^2 \otimes e_0 Q^{-1} e_0^3,$$

and, finally, *the second fundamental form*:

$$J_{ab} = \int_0^1\int_0^1 a(x)\frac{\partial n(x)}{\partial q(y)} b(y)\,dxdy$$
$$= -2\left(\int_0^1 e_0^4\right)^{-1/2}\int_0^1\int_0^1 a(x)e_0(x)Q_{xy}^{-1}e_0(y)b(y)\,dxdy$$

for directions a and b tangent to M at q, $\int_0^1 ae_0^2 = \int_0^1 be_0^2 = 0$. This makes the second portion of $\partial n/\partial q$ drop out. Now let a and b form a unit perpendicular frame: $\int_0^1 a^2 = \int_0^1 b^2 = 1$, $\int_0^1 ab = 0$. They define a 2-dimensional slice of M with curvature

$$K_{ab} = J_{aa}J_{bb} - J_{ab}^2.$$

This number is necessarily positive, J being strictly negative on the tangent space:

$$J_{cc} = -2\left(\int_0^1 e_0^4\right)^{-1/2} \sum_{n=1}^{\infty} \lambda_n^{-1}(e_n, e_0c)^2 < 0 \quad \text{if } c \neq 0.$$

3. The allied operator

The form J is closely connected to the so-called *allied operator* $\bar{Q}$. Introduce the new scale

$$\bar{x} = \left(\int_0^1 e_0\right)^{-1} \int_0^x e_0,$$

and view

$$\bar{e}_0 = \left(\int_0^1 e_0^4 dx\right)^{-1/2} \left(\int_0^1 e_0 dx\right)^{1/2} e_0^{3/2}$$

as a function of $0 \leqslant \bar{x} < 1$, noticing that $\int_0^1 (\bar{e}_0)^2\, d\bar{x} = 1$. $\bar{Q}$ is now defined to be the Hill's operator with ground state $\bar{e}_0$ relative to the scale $\bar{x}$, and with the notation (the discrepancy between this notation and $\bar{e}_0$ will not prove troublesome):

$$\bar{f}(\bar{x}) = \left(\int_0^1 e_0\, dx\right)^2 e_0^{-1/2}(x) f(x),$$

direct computation provides the identity

$$\bar{Q} e_0^{1/2} Q^{-1} e_0 f = \bar{f},$$

in which the necessary condition of perpendicularity $[\int_0^1 e_0^2 f\, dx = 0]$ for the existence of $Q^{-1} e_0 f$ is satisfied if and only if $\int_0^1 \bar{e}_0 \bar{f}\, d\bar{x}$ also vanishes. Then $\bar{Q}^{-1}\bar{f}$ exists, and the upshot is that $e_0^{1/2} Q^{-1} e_0 f = \bar{Q}^{-1}\bar{f}$. This permits a simplified expression of the second fundamental form:

$$J_{ab} = -2\left(\int_0^1 e_0^4\right)^{-1/2} \left(\int_0^1 e_0\right)^{-1} \int \bar{a} \bar{Q}^{-1} \bar{b}\, d\bar{x}.$$

Notice

$$\int_0^1 \bar{a}\bar{b}\,d\bar{x} = \left(\int_0^1 e_0\right)^2 \int_0^1 abe_0^{-1}\left(\int_0^1 e_0\right)^{-1} e_0\,dx = \left(\int_0^1 e_0\right)^3 \int_0^1 abdx,$$

so that $ab \to \bar{a}\bar{b}$ maintains perpendicularity. The point of all this computation is

Corollary 1. *The principal curvatures of M at q, i.e., the eigenvalues of the second fundamental form J, are simply*

$$-2\left(\int_0^1 e_0^4\right)^{-1/2}\left(\int_0^1 e_0\right)^{-1}\left(\int_0^1 e_0\right)^3 \times \textit{the eigenvalues of } \bar{Q}^{-1}$$

$$= -2\left(\int_0^1 e_0^4\right)^{-1/2}\left(\int_0^1 e_0\right)^2 \times \textit{the reciprocals of the excited eigenvalues of } \bar{Q}.$$

The latter are written $0 < \bar{\lambda}_1 \leq \bar{\lambda}_2 \leq \bar{\lambda}_3 \leq \bar{\lambda}_4 <$ etc.

Corollary 2. *The maximal curvature of a 2-dimensional slice of M at q is*

$$m = 4\left(\int_0^1 e_0\right)^4\left(\int_0^1 e_0^4\right)^{-1} \times (\bar{\lambda}_1\bar{\lambda}_2)^{-1}.$$

Corollary 3. *The total curvature of M at q is*

$$k = 4\left(\int_0^1 e_0\right)^4\left(\int_0^1 e_0^4\right)^{-1} \times \sum_{1\leq i<j} (\bar{\lambda}_i\bar{\lambda}_j)^{-1}.$$

The rest of the paper is devoted to the investigation of these numbers.

Proof of Corollary 2. The curvature of the general slice may be expressed as the product of $4(\int_0^1 e_0)^4(\int_0^1 e_0^4)^{-1}$ and

$$\sum \frac{a_i^2}{\bar{\lambda}_i}\sum\frac{b_j^2}{\bar{\lambda}_j} - \left(\sum\frac{a_ib_i}{\bar{\lambda}_i}\right)^2 = \sum_{i<j}\frac{(a_ib_j - a_jb_i)^2}{\bar{\lambda}_i\bar{\lambda}_j},$$

with $\sum a_i^2 = \sum b_j^2 = 1$ and $\sum a_ib_i = 0$. The final sum is over-estimated by the product of $(\bar{\lambda}_1\bar{\lambda}_2)^{-1}$ and $\sum_{i<j}(a_ib_j - a_jb_i)^2 = 1$.

Amplification 1. Let $\bar{e}$ be an excited eigenfunction of $\bar{Q}$ with eigenvalue $\bar{\lambda}$. Then $e = (\int_0^1 e_0^{-2})e_0^{1/2}\bar{e}$ satisfies $(\int_0^1 e_0)^2 e_0^{-1}Qe_0^{-1}e = \bar{\lambda}e$ and vice versa. Now Q can be expressed as $-e_0^{-1}De_0^2De_0^{-1}$, so $\bar{\lambda}$ is an eigenvalue of $-(\int_0^1 e_0)^2e_0^{-2}De_0^2De_0^{-2}$ which is similar to $-(\int_0^1 e_0)^2e_0^{-4}De_0^2D$ and so also to $-(\int_0^1 e_0)^2(\int_0^1 e_0^{-2})^{-2}e_0^{-6}D^2$, in which the differentiation is now with regard to the new scale $(\int_0^1 e_0^{-2})^{-1}\int_0^x e_0^{-2}$. This remark will be helpful in §5.

Amplification 2. $\bar{Q}$ can be any Hill's operator with $\bar{\lambda}_0 = 0$.

Proof. Let $\bar{Q}$ be the general Hill's operator relative to the fixed scale $\bar{x}$, and $\bar{e}_0$ its ground state; it is required to prove that $\bar{Q}$ is allied to some Q. Define $e_0^{3/2}(x) = a\bar{e}_0(\bar{x})$ with a new scale x specified by $dx = be_0^{-1}d\bar{x} = c(\bar{e}_0)^{-3/2}d\bar{x}$, the constants a, b, c being chosen to make $x = 1$ at the end and $\int_0^1 e_0^2\,dx = 1$. This can be done:

$$1 = x(1) = b\int_0^1 e_0^{-1}dx = ba^{-2/3}\int_0^1 (\bar{e}_0)^{-3/2}\,d\bar{x},$$

$$c = ba^{-2/3}, \quad 1 = \int_0^1 e_0^2\,dx = a^{2/3}b\int_0^1 (\bar{e}_0)^{3/2}\,d\bar{x}.$$

Then $\bar{Q}$ is allied to the Hill's operator Q with ground state e_0 relative to the scale x; indeed, $\bar{e}_0 = (\int_0^1 e_0^4)^{-1/2}(\int_0^1 e_0)^{1/2}e_0^{3/2}$, as it should be, in view of

$$1 = \int_0^1 (\bar{e}_0)^2\,d\bar{x} = \frac{1}{a^2b}\int_0^1 e_0^4\,dx, \quad b = b\int_0^1 d\bar{x} = \int_0^1 e_0\,dx.$$

This fact will be helpful in §4.

4. Maximal Curvature

The purpose of this section is to prove that the maximal curvature m can be made as large as you please; in the next section, it is shown that the total curvature can be made as small as you please, so *anything can happen*.

Proof. m can be expressed as the reciprocal of $(\int_0^1(\bar{e}_0)^{-2/3}\,d\bar{x})^3 \times \bar{\lambda}_1\bar{\lambda}_2$ in the notation of §3, and as $\bar{Q}$ can be any Hill's operator at all, so it is required to prove that $(\int_0^1 e_0^{-2/3}\,dx)^3\lambda_1\lambda_2$ can be made small by choice of Q. Now

$$\lambda_1 = \int_0^1 e_1 Q e_1 = \int_0^1 \left|\left(\frac{e_1}{e_0}\right)'\right|^2 e_0^2$$

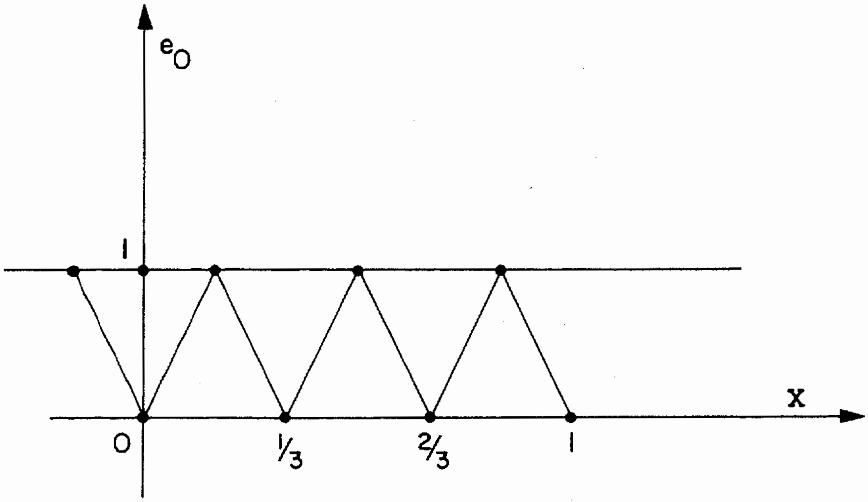
can be expressed as the minimum of the ratio of $\int_0^1(f')^2e_0^2$ to $\int_0^1 f^2e_0^2$ for f of class C_1^∞ with $\int_0^1 fe_0^2 = 0$; moreover, $\lambda_1 = \lambda_2$ if q is of period $1/3$, Borg [1], so it suffices to make

$$I = \left(\int_0^1 e_0^{-2/3}\right)^{3/2}\int_0^1 (f')^2e_0^2\left(\int_0^1 f^2e_0^2\right)^{-1}$$

small for even e_0 of period $1/3$ and odd f. Choose e_0 to approximate the saw-tooth function of Fig. 1 and let the odd function f be $\pm e_0^p$. Then I is closely approximated by a fixed multiple of

$$\frac{\int_0^{1/6}p^2x^{2p-2}x^2\,dx}{\int_0^{1/6}x^{2p}x^2\,dx} = 36p^2\frac{2p+3}{2p+1}$$

and is small for $p = 0+$.



5. Total curvature

The total curvature k can be expressed in the following compact form:

$$k = 4\left(\int_0^1 e_0^{-2}\right)^2\left(\int_0^1 e_0^4\right)^{-2}\int\int\int e_0^4(x_1)e_0^4(x_2)e_0^4(x_3)$$
$$\times\int_{x_1^*}^{x_2^*} e_0^{-2}\int_{x_2^*}^{x_3^*} e_0^{-2}\int_{x_3^*}^{x_1^*} e_0^{-2}\, d^3x,$$

mentioned in §1.

Proof. The author owes the idea of this proof to a remark of G. Segal. The periodic spectrum of $\overline{Q}$ may be described [2] as the roots of $\overline{\Delta}(\lambda) = +2$, $\overline{\Delta}$ being the discriminant of $\overline{Q}$. $\overline{\Delta}$ is now expressed with the aid of the similar operator of Amplification 1 of §3:

$$-D_b D_a, \quad da = \left(\int_0^1 e_0^{-2}\right)^{-1} e_0^{-2}dx, \quad db = \left(\int_0^1 e_0\right)^{-2}\left(\int_0^1 e_0^{-2}\right)e_0^4\, dx.$$

The formula is

$$\overline{\Delta}(\lambda) = [y_1(1, \lambda) + y_2'(1, \lambda)].$$

The prime signifies differentiation with respect to a,

$$y_1(x, \lambda) = 1 + \lambda\int_0^x da\int_0^{x_1} db + \lambda^2\int_0^x da\int_0^{x_1} db\int_0^{x_2} da\int_0^{x_3} db + \text{etc.},$$

$$y_2(x, \lambda) = a(x) + \lambda\int_0^x da\int_0^{x_1} a\, db$$
$$+\lambda^2\int_0^x da\int_0^{x_1} db\int_0^{x_2} da\int_0^{x_3} a\, db + \text{etc.},$$

and from the product $c\lambda\prod_{n=1}^{\infty}(1-\lambda/\bar{\lambda}_n)$ for $\bar{\Delta}(\lambda)-2$ is obtained

$$6\sum_{i<j}\left(\bar{\lambda}_i\bar{\lambda}_j\right)^{-1}=\left(\frac{d\bar{\Delta}}{d\lambda}\right)^{-1}\frac{d^3\bar{\Delta}}{d\lambda^3}\quad\text{evaluated at } \lambda=0$$

$$=2\left(\int_0^1 e_0\right)^2\left(\int_0^1 e_0^{-2}\right)^{-1}\left(\int e_0^4\right)^{-1}\times 3\left(\int_0^1 e_0\right)^{-6}$$

$$\times\left[\int_0^1 e_0^{-2}\int_0^{x_1}e_0^4\int_0^{x_2}e_0^{-2}\int_0^{x_3}e_0^4\int_0^{x_4}e_0^{-2}\int_0^{x_5}e_0^4\,d^6x\right.$$

$$\left.+\int_0^1 e_0^4\int_0^{x_1}e_0^{-2}\int_0^{x_2}e_0^4\int_0^{x_3}e_0^{-2}\int_0^{x_4}e_0^4\int_0^{x_5}e_0^{-2}\,d^6x\right].$$

This expression is inserted into

$$k=4\left(\int_0^1 e_0\right)^4\left(\int_0^1 e_0^4\right)^{-1}\times\sum_{i<j}\left(\bar{\lambda}_i\bar{\lambda}_j\right)^{-1},$$

and the result is reduced to the stated form by exchange of integrals.

The formula is applied to confirm that k can be made as small as one pleases: it suffices to let e_0 approximate x^p with $1/2>p>-1/4$ and to estimate

$$k\leqslant 24(1+4p)(1-2p)^{-5}2^{1-2p}\qquad \text{as } p\downarrow -1/4.$$

References

[1] G. Borg, *Eine Umkehrung der Sturm-Liouvillieschen Eigenwertaufgabe*, Acta Math. **78** (1945) 1–96.

[2] W. Magnus & W. Winkler, *Hill's equation*, Wiley-Interscience, New York, 1966.

NEW YORK UNIVERSITY